# The cumulant expansion in renormalization group transformations: the convergence question and the dependence upon cell size



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# The cumulant expansion in renormalization group transformations: the convergence question and the dependence upon cell size

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**Abstract.** The Niemeijer–van Leeuwen cumulant expansion for calculation of renormalized interactions is carried to third order for the Ising model on the triangular lattice. Kadanoff cells containing three spins are used. Five types of pair interactions and three types of four-spin interactions enter the calculation.

The renormalization equations have one non-trivial fixed point. The accuracy of the result for the corresponding relevant eigenvalues is, however, less than in second order. The idea that the cumulant expansion is an asymptotic expansion has thus been given numerical support.

## 1. Introduction

An attractive computational method to evaluate renormalization recursion relations for discrete spin systems is the cumulant expansion procedure of Niemeijer and van Leeuwen (1973, 1974). In this renormalization scheme one divides the spins into cells, each containing $m$ spins $s_i = \pm 1$, and associates with each cell a cell spin $s'$, usually defined by the majority rule

$$s' = \operatorname{sgn}\left(\sum_{i=1}^{m} s_i\right), \tag{1}$$

summed over the spins in the cell. The cell spins should be located on a lattice isomorphic with the original lattice. In the Niemeijer–van Leeuwen scheme one computes the effective interactions between the cell spins by dividing the spin Hamiltonian $H(s)$ into an intracell part $H_0(s)$ and an intercell part $V(s)$, treating the latter as a perturbation. This leads straightforwardly to a cumulant expansion for the cell spin Hamiltonian $H'(s')$.

Two basic questions naturally arise. Firstly, what is the nature of the cumulant expansion? In particular, does it converge? Secondly, what is the importance, if any, of the cell size $m$? In an exact renormalization calculation results should be independent of $m$, but the question is which choice of $m$ is to be preferred in an approximate calculation of the cumulant expansion to second order, say.

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The answers to these questions are not known, although it has been speculated (Niemeijer and van Leeuwen 1975, Hsu *et al* 1975) that the cumulant expansion yields merely an asymptotic series, such that one can terminate the expansion at an optimal order $o_m$. Moreover, the tentative idea is that both $o_m$ and the accuracy of the result increase with increasing cell size $m$. In this paper we present numerical evidence in support of this conjecture.

According to the conjecture the lack of convergence should be most easily detected for small cells, and we choose to work with *three*-spin cells on the triangular lattice (as in the original Niemeijer–van Leeuwen calculation). The additional advantage is that one can compare with recent results on the same lattice with *seven*-spin cells (Sudbø and Hemmer 1976). In that calculation, as well as in the present one, the cumulant expansion is carried to third order.

## 2. The renormalized coupling constants

The renormalized Hamiltonian is generated by the cumulant expansion

$$H'(s') = \ln\langle \exp V\rangle_0 = \langle V\rangle_0 + \tfrac{1}{2}(\langle V^2\rangle_0 - \langle V\rangle_0^2) + \tfrac{1}{3!}(\langle V^3\rangle_0 - 3\langle V^2\rangle_0\langle V\rangle_0 + 2\langle V\rangle_0^3) + O(V^4). \tag{2}$$

The averages $\langle\ \rangle_0$ are with respect to the Boltzmann factor $\exp(H_0)$, and are averages over all spin configurations $\{s\}$ compatible with a given cell-spin configuration $\{s'\}$. A factor $(-\beta)$ is included in the Hamiltonian and in the coupling constants. We assume zero magnetic field. A constant term has been omitted in (2). We refer to Niemeijer and van Leeuwen (1974) for details of the procedure.

The eight coupling constants required to generate the full third-order recursion relations are shown in figure 1. They are classified as first-, second- and third-order quantities according to the lowest order of the cumulant expansion in which they are generated from a pure nearest-neighbour pair spin interaction.

The evaluation of (2) to third order yields the following non-linear equations for the renormalized couplings $K'_n$ between the cell spins:

$$\begin{aligned} K'_{21} = {} & 2a^2K_{21} + 3a^2K_{22} + 2a^2K_{23} + (4a^2 + 4a^2b - 8a^4)K_{21}^2 \\ & + 2a^2K_{24} + (ac + 2a^2b)K_{41} + (2ac + 4a^2b)K_{42} + 2a^2bK_{43} \\ & + (8a^2 + 16a^2b - 24a^4)K_{21}K_{22} + (12a^2 + 12a^2b - 24a^4)K_{21}K_{23} \\ & + (22a^2/3 + 4ac + 4a^2b + 14a^2b^2 - 40a^4 - 104a^4b + 344a^6/3)K_{21}^3 \end{aligned} \tag{3a}$$

$$\begin{aligned} K'_{22} = {} & a^2K_{23} + (a^2 + 7a^2b - 8a^4)K_{21}^2 + 2a^2K_{24} + 3a^2K_{25} + 2a^2bK_{42} \\ & + (8a^2 + 16a^2b - 24a^4)K_{21}K_{22} + (6a^2 + 10a^2b - 16a^4)K_{21}K_{23} \\ & + (4a^2 + 24a^2b + 20a^2b^2 - 32a^4 - 64a^4b + 48a^6)K_{21}^3 \end{aligned} \tag{3b}$$

$$\begin{aligned} K'_{23} = {} & (4a^2b - 4a^4)K_{21}^2 + 2a^2K_{24} + (4a^2 + 8a^2b - 12a^4)K_{21}K_{22} \\ & + (4a^2 + 12a^2b - 16a^4)K_{21}K_{23} \\ & + (2a^2 + 20a^2b + 26a^2b^2 - 24a^4 - 72a^4b + 48a^6)K_{21}^3 \end{aligned} \tag{3c}$$

$$K'_{24} = (4a^2b - 4a^4)K_{21}K_{23} + (20a^2b^2 + 4a^2b - 4a^4 - 44a^4b + 24a^6)K_{21}^3 \tag{3d}$$

$$K'_{25} = (8a^2b^2 - 16a^4b + 8a^6)K_{21}^3 \tag{3e}$$

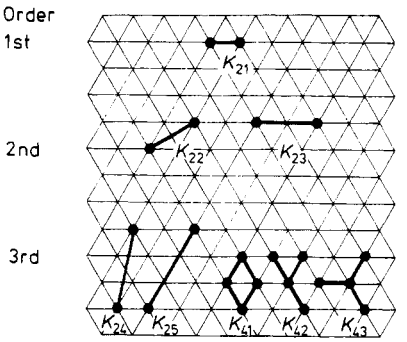


**Figure 1.** First, second, and third order interactions in zero field.

$$K'_{41} = 2a^4 K_{42} + (2a^3 c - 4a^4 - 30a^4 b + 32a^6) K_{21}^3 \tag{3f}$$

$$K'_{42} = (3a^3 c - 19a^4 b + 16a^6) K_{21}^3 \tag{3g}$$

$$K'_{43} = (5a^3 c - 21a^4 b + 16a^6) K_{21}^3. \tag{3h}$$

We have introduced the following abbreviations for averages of one, two or three spins within one cell with cell spin $s'$:

$$a = \langle s_1 \rangle_0 / s' = \frac{\exp(4K_{21}) + 1}{\exp(4K_{21}) + 3} \tag{4}$$

$$b = \langle s_1 s_2 \rangle_0 = 2a - 1 \tag{5}$$

$$c = \langle s_1 s_2 s_3 \rangle_0 / s' = 3a - 2. \tag{6}$$

The second-order terms in (3) agree with the Niemeijer–van Leeuwen expressions.

## 3. Fixed point and eigenvalue for three-spin cells

The renormalization equations (3) have just one non-trivial fixed point, namely

$$\begin{aligned} &K^*_{21} = 0{\cdot}520559; \quad && K^*_{22} = -0{\cdot}038370; \quad && K^*_{23} = -0{\cdot}051555; \\ &K^*_{24} = 0{\cdot}000310; \quad && K^*_{25} = 0{\cdot}000820; \quad && K^*_{41} = -0{\cdot}019164; \\ &K^*_{42} = 0{\cdot}018077; \quad && K^*_{43} = 0{\cdot}007888. \end{aligned} \tag{7}$$

Linearizing the renormalization equations around this fixed point we find merely one relevant eigenvalue $\lambda_t$ of the transformation matrix

$$T_{mn} = (\partial K'_m / \partial K_n)_{K=K^*}. \tag{8}$$

The value of $\lambda_t$ is given in table 1 together with the results of first and second order cumulant expansion (Niemeijer and van Leeuwen 1974). The exact value of the eigenvalue is $\lambda_t = \sqrt{3}$. The Widom homogeneity exponent

$$y_t = \ln \lambda_t / \ln \sqrt{3}$$

**Table 1.** Thermal eigenvalue and homogeneity index from successive orders in the cumulant expansion.

| | Three-spin cells | | Seven-spin cells | |
|---|---|---|---|---|
| Approximation | $\lambda_t$ | $y_t$ | $\lambda_t$ | $y_t$ |
| First order | 1·624 | 0·883 | 2·444 | 0·919 |
| Second order | 1·773 | 1·042 | 2·986 | 1·124 |
| Third order | 1·877 | 1·146 | 2·838 | 1·072 |
| Exact | 1·732 | 1·000 | 2·646 | 1·000 |

is also recorded in table 1, together with corresponding results for seven-spin cells (Sudbø and Hemmer 1976).

The intersection of the critical surface in the eight-dimensional parameter space with the $K_{21}$ axis yields the inverse critical temperature $K_{21}^c$ for the model with only nearest-neighbour interactions. The results can be compared with the known exact value $K_{21}^c = \frac{1}{4} \ln 3 = 0{\cdot}2747\ldots$. The numerical values for the present three-spin cell transformation are recorded in table 2, together with the corresponding results for seven-spin cells (Sudbø and Hemmer 1976).

**Table 2.** Critical coupling constant for the model with nearest-neighbour interactions.

| | Three-spin cells | Seven-spin cells |
|---|---|---|
| Approximation | $K_{21}^c$ | $K_{21}^c$ |
| First order | 0·3356 | 0·3003 |
| Second order | 0·2575 | 0·2647 |
| Third order | 0·3009 | 0·2752 |
| Exact | 0·2747 | 0·2747 |

## 4. Discussion

One sees from the result given for three-spin cells in table 1 that inclusion of the third order on the contrary does *not* improve the result for the thermal eigenvalue. The same is true for the critical temperature calculation recorded in table 2. The second-order critical coupling constant is off by 6·3%, while the third-order value deviates 9·5% from the exact value. (With the larger cell the corresponding deviations are merely 3·6% and 0·2%.)

Although based upon scanty evidence, this supports the conjecture that the cumulant expansion is an asymptotic expansion, with an optimal order $o_3$ for three-spin cells equal to 2.

For the larger cell, on the other hand, is it clear from the result given in table 1 that $o_7 \geqslant 3$. Thus the conjecture that $o_m$ is increasing with $m$ has been given numerical support. The rationale behind this part of the conjecture is of course that the intercell part $V$, i.e. the perturbation, constitutes a smaller part of the Hamiltonian when the cell size is large.

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